ON THE KÄHLER STRUCTURES OVER QUOT SCHEMES, II

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Abstract. Let X be a compact connected Riemann surface of genus g, with $g \geq 2$, and let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X. Fix positive integers r and d and let $\mathcal{Q}(r,d)$ be the Quot scheme parametrizing all torsion coherent quotients of $\mathcal{O}_X^{\oplus r}$ of degree d. We prove that $\mathcal{Q}(r,d)$ does not admit a Kähler metric whose holomorphic bisectional curvatures are all nonnegative.

1. Introduction

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. For any positive integer d, let $S^d(X)$ denote the d-fold symmetric product of X. The main theorem of [BoR] says the following (see [BoR, Theorem 1.1]): If $d \leq 2(g-1)$, then $S^d(X)$ does not admit any Kähler metric for which all the holomorphic bisectional curvatures are nonnegative.

A natural generalization of the symmetric product $S^d(X)$ is the Quot scheme $\mathcal{Q}(r,d)$ that parametrizes all torsion coherent quotients of $\mathcal{O}_X^{\oplus r}$ of degree d. So, $\mathcal{Q}(1,d) = S^d(X)$. These Quot schemes arise naturally in the study of vector bundles on curves (see [BGL]). They also appear in mathematical physics as moduli spaces of vortices (cf. [Ba], [BiR], [BoR]).

Our aim here is to prove the following (see Theorem 3.1):

The complex manifold $\mathcal{Q}(r,d)$ does not admit any Kähler metric such that all the holomorphic bisectional curvatures are nonnegative.

In [BS], this was proved under the assumption that $d \leq 2(g-1)$.

2. The Albanese map for $\mathcal{Q}$

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. The sheaf of germs of holomorphic functions on X will be denoted by $\mathcal{O}_X$. Fix positive integers r and d. Let

$$\mathcal{Q} := \mathcal{Q}(r,d)$$

be the Quot scheme that parametrizes all torsion coherent quotients of $\mathcal{O}_X^{\oplus r}$ of degree d. In other words, $\mathcal{Q}$ parametrizes all coherent subsheaves of $\mathcal{O}_X^{\oplus r}$ of rank r and degree $-d$.

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The group of permutations of $\{1\,,\cdots,d\}$ will be denoted by P_d. Let $S^d(X) := X^d/P_d$ be the symmetric product of X. The Quot scheme $\mathcal{Q}(1,d)$ is identified with $S^d(X)$ by sending any quotient $Q' \in \mathcal{Q}(1,d)$ of $\mathcal{O}_X$ to the scheme-theoretic support of Q'. Take any quotient $Q \in \mathcal{Q} = \mathcal{Q}(r,d)$. Consider the corresponding short exact sequence

$$0 \longrightarrow \mathcal{K}_Q \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow Q \longrightarrow 0$$

on X. Let

$$0 \longrightarrow \bigwedge^r \mathcal{K}_Q \longrightarrow \bigwedge^r \mathcal{O}_X^{\oplus r} = \mathcal{O}_X \longrightarrow (\bigwedge^r \mathcal{O}_X^{\oplus r})/(\bigwedge^r \mathcal{K}_Q) := Q' \longrightarrow 0$$

be the short exact sequence obtained from it by taking r-th exterior product. Sending any $Q \in \mathcal{Q}$ to $Q' \in \mathcal{Q}(1,d)$ constructed as above from Q we get a morphism

$$f\,:\,\mathcal{Q} \longrightarrow S^d(X) = \mathcal{Q}(1,d)\,. \tag{2.1}$$

Lemma 2.1. *Take any point $z \in S^d(X)$. There is a sequence of holomorphic maps ending at the point z,*

$$Y_d^z \xrightarrow{p_d} Y_{d-1}^z \xrightarrow{p_{d-1}} \cdots \xrightarrow{p_2} Y_1^z \xrightarrow{p_1} Y_0^z = \{z\}\,,$$

such that

- *all the fibers of each p_i are isomorphic to $\mathbb{CP}^{r-1}$, and*
- *there is a surjective holomorphic map from Y_d^z to the fiber $f^{-1}(z)$.*

Proof. Fix a point

$$(x_1\,,\cdots,x_d) \in X^d$$

that projects to z under the quotient map $X^d \longrightarrow X^d/P_d = S^d(X)$. For any integer $i \in [1\,,d]$, let Y_i^z be the space of all filtrations of coherent analytic subsheaves

$$W_i \subset W_{i-1} \subset \cdots \subset W_1 \subset W_0 = \mathcal{O}_X^{\oplus r}\,, \tag{2.2}$$

where W_j/W_{j+1}, $0 \leq j \leq i-1$, is a torsion sheaf supported at x_{j+1} such that the dimension of the stalk of W_j/W_{j+1} at x_{j+1} is one. Therefore, W_{j+1} is a holomorphic vector bundle of rank r and degree $-j-1$. In particular, W_i is a holomorphic vector bundle of rank r and degree $-i$. We have the natural forgetful map

$$p_i\,:\,Y_i^z \longrightarrow Y_{i-1}^z$$

that forgets the smallest subsheaf in the filtration (it forgets W_i in (2.2)). The inverse image, under the projection p_i, of a point

$$W_{i-1} \subset \cdots \subset W_1 \subset W_0 = \mathcal{O}_X^{\oplus r}$$

of Y_{i-1}^z is the projective space $\mathbb{P}(W_{i-1})_{x_i}$ that parametrizes all hyperplanes in the fiber of the vector bundle W_{i-1} over the point x_i. In particular, the map p_i makes Y_i^z a projective bundle over Y_{i-1}^z of relative dimension $r-1$.

Sending any point

$$W_d \subset W_{d-1} \subset \cdots \subset W_1 \subset W_0 = \mathcal{O}_X^{\oplus r}\,,$$

of Y^z_d to the subsheaf $W_d \subset \mathcal{O}_X^{\oplus r}$ we get a holomorphic map

$$Y^z_d \longrightarrow f^{-1}(z)\,.$$

This map is clearly surjective. $\square$

Let

$$\varphi \,:\, S^d(X) \longrightarrow \mathrm{Pic}^d(X) \tag{2.3}$$

be the morphism that sends a divisor to the corresponding holomorphic line bundle on X.

Corollary 2.2. *The composition $\varphi\circ f \,:\, \mathcal{Q} \longrightarrow \mathrm{Pic}^d(X)$ is the Albanese morphism for the variety $\mathcal{Q}$.*

Proof. Since there is no nonconstant morphism from a rational curve to an abelian variety, from Lemma 2.1 it follows immediately that the Albanese morphism for $\mathcal{Q}$ factors through f: Take any holomorphic line bundle $L \in \mathrm{Pic}^d(X)$. Note that the fiber $\varphi^{-1}(L)$ is the projective space $\mathbb{P}H^0(X,\, L)^*$. Therefore, the Albanese morphism for $\mathcal{Q}$ factors through $\varphi\circ f$. $\square$

3. Holomorphic bisectional curvatures

We continue with the notation of Section 2.

Theorem 3.1. *The complex manifold $\mathcal{Q}$ does not admit any Kähler metric such that all the holomorphic bisectional curvatures are nonnegative.*

Proof. Assume that $\mathcal{Q}$ admits a Kähler metric g_Q with nonnegative holomorphic bisectional curvatures. Let ω denote the Kähler form of g_Q.

Let

$$\rho \,:\, \widetilde{\mathcal{Q}} \longrightarrow \mathcal{Q} \tag{3.1}$$

be the universal cover. Then, by Mok's uniformization theorem for compact Kähler manifolds with nonnegative bisectional curvature [Mo, p. 179, Main Theorem], the Kähler manifold $(\widetilde{\mathcal{Q}}\,,\rho^*g_Q)$ decomposes as a Cartesian product

$$(\widetilde{\mathcal{Q}}\,,\rho^*g) \,=\, (\mathbb{C}^N\,,g_0)\times(\prod_{i=1}^{k}(\mathbb{C}P^{r_i}\,,g_i))\times(\prod_{j=1}^{\ell}((H_j\,,h_j))$$

of Kähler manifolds for some nonnegative integers $r_1\,,\cdots\,,r_k,\,,\ell$; here $(\mathbb{C}^N\,,g_0)$ is the Euclidean space with its standard complex structure and metric, g_i are Kähler metrics of nonnegative bisectional curvature on $\mathbb{C}P^{r_i}$, and h_j are the standard symmetric Kähler metrics on certain Hermitian symmetric spaces H_j of compact type. In what follows we do not need the specific information on the non-Euclidean factors in the decomposition. We write

$$(\widetilde{\mathcal{Q}}\,,\rho^*g_Q) \,=\, (\mathbb{C}^N\,,g_0)\times(H\,,g')\,,$$

where $(H, g') = (\prod_{i=1}^{k}(\mathbb{C}P^{r_i}, g_i)) \times (\prod_{j=1}^{\ell}((H_j, h_j))$.

Let

$$q_1 : \mathbb{C}^N \times H \longrightarrow \mathbb{C}^N$$

be the projection to the first factor. Let

$$\gamma \in \mathrm{Gal}(\rho)$$

be a deck transformation for the covering ρ in (3.1). Since there is no nonconstant holomorphic map from H to $\mathbb{C}^N$, there is a holomorphic isometry $\gamma' \in \mathrm{Aut}(\mathbb{C}^N)$ such that

$$q_1 \circ \gamma = \gamma' \circ q_1 .$$

Therefore, q_1 descends to a holomorphic submersion of the form

$$\beta : \mathcal{Q} \longrightarrow Y \tag{3.2}$$

such that Y is (regularly) covered by $\mathbb{C}^N$; so $\mathbb{C}^N$ is a universal cover of Y. More precisely, there is a Kähler form preserving covering map

$$\rho' : \mathbb{C}^N \longrightarrow Y$$

such that $\rho' \circ q_1 = \beta \circ \rho$. We note that β makes $\mathcal{Q}$ a holomorphic fiber bundle over Y with fibers isomorphic to H.

Consider the C^∞ distribution

$$\mathcal{S} \subset T\mathcal{Q}$$

on $\mathcal{Q}$ obtained by taking the orthogonal complement, with respect to ω, of the relative tangent bundle for the projection β in (3.2). The pulled back distribution

$$\rho^\star\mathcal{S} = (d\rho)^{-1}(\mathcal{S}) \subset T\widetilde{\mathcal{Q}}$$

coincides with the relative tangent bundle for the projection $q_2 : \mathbb{C}^N \times H \longrightarrow H$ to the second factor; here $d\rho : T\widetilde{\mathcal{Q}} \longrightarrow T\mathcal{Q}$ is the differential of ρ. In particular, $\rho^\star\mathcal{S}$ is an integrable holomorphic distribution. In other words, $\rho^\star\mathcal{S}$ defines a flat holomorphic connection on the fiber bundle

$$q_1 : \mathbb{C}^N \times H \longrightarrow \mathbb{C}^N .$$

This implies that $\mathcal{S}$ is a flat holomorphic connection on the fiber bundle $\beta : \mathcal{Q} \longrightarrow Y$. Let

$$\nabla \tag{3.3}$$

denote this flat holomorphic connection on $\beta : \mathcal{Q} \longrightarrow Y$.

We now need the following proposition.

Proposition 3.2. *The fibers of the map β coincides with the fibers of the map $\varphi \circ f$. In particular, there is a biholomorphism*

$$\alpha : Y \longrightarrow \mathrm{image}(\varphi \circ f)$$

such that $\alpha \circ \beta = \varphi \circ f$.

Proof of Proposition 3.2. Take a point $y \in Y$. Consider the restriction of the map $\varphi \circ f$ to $\beta^{-1}(y)$. Since $\beta^{-1}(y) = H$ is simply connected, this map

$$(\varphi \circ f)|_{\beta^{-1}(y)} : \beta^{-1}(y) \longrightarrow \mathrm{Pic}^d(X)$$

lifts to a map

$$\beta^{-1}(y) \longrightarrow \widetilde{\mathrm{Pic}}^d(X),$$

where $\widetilde{\mathrm{Pic}}^d(X)$ is the universal cover of $\mathrm{Pic}^d(X)$. But $\widetilde{\mathrm{Pic}}^d(X)$ is isomorphic to $\mathbb{C}^g$, and hence any holomorphic map to it from $\beta^{-1}(y)$ is a constant map.

Conversely, take any point $L \in \varphi \circ f(\mathcal{Q})$. We will show that the restriction of the map β to the fiber $(\varphi \circ f)^{-1}(L)$ is a constant map. For this we first show that there is no nonconstant holomorphic map from $\mathbb{CP}^1$ to Y. To prove this note that $\mathbb{CP}^1$ is simply connected and the universal over of Y is $\mathbb{C}^N$. Therefore, any holomorphic map from $\mathbb{CP}^1$ to Y lifts to a holomorphic map from $\mathbb{CP}^1$ to $\mathbb{C}^N$, but there are no such nonconstant holomorphic maps.

Since there is no nonconstant holomorphic map from $\mathbb{CP}^1$ to Y, from Lemma 2.1 it follows that the map

$$\beta|_{(\varphi\circ f)^{-1}(L)} : (\varphi \circ f)^{-1}(L) \longrightarrow Y$$

factors through $(\varphi \circ f)^{-1}(L) \xrightarrow{f} \varphi^{-1}(L)$. As noted in the proof of Corollary 2.2, the fiber $\varphi^{-1}(L)$ is a projective space. Hence the restriction of β to the fiber $(\varphi \circ f)^{-1}(L)$ is a constant map. This completes the proof of the proposition. $\square$

Continuing with the proof of the theorem, since $g \geq 2$, we know that $\mathcal{Q}(r, d)$ does not admit any Kähler metric with nonnegative holomorphic bisectional curvature if $d < g$ [BS, Corollary 4.1]. Therefore, we assume that $d \geq g$. Consequently, the map φ is surjective. Since f is surjective, this implies that $\varphi \circ f$ is surjective. Now from Proposition 3.2 we know that Y is an abelian variety.

Choose a point of Y to make it a group. The universal covering map $\mathbb{C}^N \longrightarrow Y$ is chosen to be a homomorphism of groups. For any point v in the universal cover $\mathbb{C}^N$ of Y, let

$$\tau_v : Y \longrightarrow Y$$

be the automorphism given by the automorphism $w \longmapsto w + v$ of $\mathbb{C}^N$. We have a one-parameter family of automorphisms Y given by τ_{tv}, $t \in [0\,,1]$, that connects τ_v with the identity map of Y.

Consider the flat holomorphic connection ∇ in (3.3). Given a point $y \in Y$, we may take parallel translation of the fiber of β over y along the path $\tau_{tv}(y)$, $t \in [0\,,1]$. Taking parallel translations for ∇ along such paths τ_{tv}, $t \in [0\,,1]$, we get holomorphic automorphisms of $\mathcal{Q}$. If v is sufficiently close to zero, then the automorphism τ_v of Y clearly does not have any fixed-point. Hence for v sufficiently close to zero, we get holomorphic automorphisms of $\mathcal{Q}$ that do not have any fixed point.

Let $\mathrm{Aut}(\mathcal{Q})$ denote the group of holomorphic automorphisms of $\mathcal{Q}$. Let $\mathrm{Aut}^0(\mathcal{Q}) \subset \mathrm{Aut}(\mathcal{Q})$ be the connected component of it containing the identity element.

The standard action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$ produces an action of $\mathrm{GL}(r, \mathbb{C})$ on the trivial vector bundle $\mathcal{O}_X^{\oplus r} = X \times \mathbb{C}^r$. The corresponding action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathcal{Q}$ factors through the quotient group $\mathrm{PGL}(r, \mathbb{C}) = \mathrm{GL}(r, \mathbb{C})/\mathbb{C}^*$. Since this action of $\mathrm{PGL}(r, \mathbb{C})$ on $\mathcal{Q}$ is effective, we have

$$\mathrm{PGL}(r, \mathbb{C}) \subset \mathrm{Aut}^0(\mathcal{Q})\,.$$

It is known that $\mathrm{PGL}(r, \mathbb{C}) = \mathrm{Aut}^0(\mathcal{Q})$ [BDH, Theorem 1.1]. Since the standard action on $\mathbb{CP}^{r-1}$ of any element $A \in \mathrm{PGL}(r, \mathbb{C})$ has a fixed point, the action of A on $\mathcal{Q}$ has a fixed point.

This contradicts our earlier construction of automorphisms of $\mathcal{Q}$ without fixed points. In view of this contradiction, we conclude that $\mathcal{Q}$ does not admit any Kähler structure such that all the holomorphic bisectional curvatures are nonnegative. $\square$

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